Math Olympians

Lower Secondary Level

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A Publication of FIG Learning Centre

<u>Preface</u>

"I hear and I forget.
I see and I remember.
I do and I understand."

-- Chinese Proverb.

This book is specially designed for students interested in participating in the Mathematics Olympiad, but even those who just have a casual interest in Mathematics will find the questions here intriguing and challenging.

The questions in the book are arranged according to topic, and the detailed solutions and workings can be found at the back of the volume.

We sincerely hope that by doing the questions in this book, students will understand and grasp the fundamental techniques required for critical Mathematical thinking.

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Contents

Topic: Number Theory

"The series of integers is obviously an invention of the human mind, a self-created tool which simplifies the ordering of certain sensory experiences." -- Albert Einstein

Q 1: Find the largest integer n such that

$n^{6030} < 2010^{2010}$.

Q 2: Given that n is a ten-digit number in the form $\overline{2009x2010y}$ where x and y can be any of the digits 0, 1, 2, …, 9. How many such numbers n are there that are divisible by 33?

Q 3: Find the last digit of $2^{2^{2009}} + 1$.

Q 4: What is the largest positive integer n for which $n^3 + 2009$ is divisible by $n + 29$?

Q 5: An even number $\overline{987a654a321a}$ is divisible by 9 but not divisible by 5. What is the digit a?

Q 6: A number is a palindrome if its digits are the same when written forward or backward. For example, the numbers 7, 11111, 302203 are palindromes. How many integers between 1 and 2009 inclusive are palindromes?

Q 7: A positive integer d is said to be strictly descending if, in its decimal representation: $d = \overline{d_m d_{m-1} \ldots d_2 d_1}$, we have $9 \geq d_m > d_{m-1} > \ldots > d_2 > d_1 \geq 0$.

For instance, 965 and 84321 are strictly descending integers. Find the number of strictly descending integers which are less then 10^4. (Note: We regard all single-digit positive integers as strictly descending.)

Q 8: A pattern of squares is made from matchsticks shown as follows:

If there are 112 matchsticks used, how many squares have been formed?

Q 9: Let **m = 76^{2009} -76.** Find the remainder when m is divided by 100.

Q 10: When 2009 bars of soap are packed into N boxes of equal size, where N is an integer strictly between 200 and 300, there are 7 extra bars remaining. Find N.

Topic: Number Theory

Q 1: 12

We need the largest n such that $(n^3)^{2010} < 2010^{2010}$. We need $n^3 < 2010$. By calculation $12^3 = 1728 < 2010 < 2197 = 13^3$. The result follows.

Q 2: 3

$33 = 3 \times 11$. So 3 must divide $2 + 9 + x + 2 + 1 + y$

Therefore, $14 + x + y = 3k_1$ for some integer k.

11 must divide $(9+2+1+y) - (2+x)$.

⇨ $10 + y - x = 11k_2$

$1 \le 11k_2 \le 19$

$k_2 = 1$

So, $10 + y - x = 11$

$y = x + 1$

$15 + 2x = 3k_1$

When $k_1 = 5$, $x = 0$ and $y = 1$

When $k_1 = 7$, $x = 3$ and $y = 4$

When $k_1 = 9$, $x = 6$ and $y = 7$

Q 3: 7

For $n \ge 2$, 2^{2^n} always ends with a 6.

This is because 2^n will end with 6 if n is a multiple of 4. 2^n is always a multiple of 4 for $n \ge 2$.

Q 4: n = 22351

$n^3 + 2009 = (n + 29)(n^2 - 29n + 841) - 22380$ (using long division)

So, if $n + 29$ divides $n^3 + 2009$, $n + 29$ must divide 22380. Thus the largest n is when $n + 29 = 22380$.

Therefore n = 22351.

Q 5: 6

The sum of the digits is divisible by 9, ie,

$9 \mid (9 + 8 + 7 + a + 6 + 5 + 4 + a + 3 + 2 + 1 + a) = 45 + 3a$

Therefore, the possible values of a are 0, 3, 6, 9. Since the number is even, a must be either 0 or 6. But the number is not divisible by 5, so a = 6.

Q 6: 119

There are 9 single-digit palindromes: 1, 2, 3, …. , 9

There are 9 double- digit palindromes: 11, 22, 33, … , 99

There are 9×10 triple- digit palindromes: 1a1, 2a2, 3a3, …, 9a9 where $a \in \{0, 1, 2, 3, …, 9\}$

There are 10 palindromes between 1000 and 1999: 1001, 1111, 1221, …, 1991

There is 1 palindrome between 2000 and 2009: 2002

Thus, the total number is 9 + 9 + 90 + 10 + 1 = 119.

Q 7: 384

There are 9 single-digit strictly descending integers. For $k > 1$, any k-element subset of {0, 1, 2, … 9} would correspond to a k -digit strictly descending integer $\overline{d_k d_{k-1} \ldots d_2 d_1}$. For example, the subset {3, 0, 9, 6} would correspond to the strictly descending integer 9630. There are $\binom{10}{k}$ such sets. Thus, the total number of strictly descending integers which are less than 10^4 is

$9 + \binom{10}{2} + \binom{10}{3} + \binom{10}{4} = 9 + \frac{10!}{2!8!} + \frac{10!}{3!7!} + \frac{10!}{4!6!} = 9 + 45 + 120 + 210 = 384$

Q 8: 37

The first 4 matchsticks give 1 square. Subsequently, every batch of 3 matchsticks give an additional square. Since 112 = 4 + (36 × 3), there are 36 + 1 = 37 squares.

Q 9: 0

$m = 76.(76^{2008} - 1)$

$= 76[(75+1)^{2008} - 1]$

$= 76\ [75^{2008} + 2008\ (75)^{2007} + \ldots + 1 - 1]$

$= 76 \times 75 \times k$ for some k.

76 × 75 = (2 × 2 × 19) × (3 × 5 × 5) which is a multiple of hundred as it has $2^2 \times 5^2$ as its factors. Thus 100 divides m.

Q 10: 286

By Assumption, 2009 – 7 = N.k for some integer k. Factorize 2002 = 2.7.11.13. Since 200 < N <300, the only possibility is N = 2 .11.13 = 286 and k = 7.

Q 11: 45

2009 is the 1005th odd number.

$(\frac{2009+1}{2} = 1005)$

If 2009 is in the group k + 1, then $1 + 2 + .. + k < 1005 \leq 1 + 2 + \ldots + (k + 1)$.

Thus,

$$\frac{k(k+1)}{2} < 1005 \leq \frac{(k+1)(k+2)}{2}$$

We get k = 44. So 2009 is in group 45 which has 45 odd integers.

Q 12: 49

Let 2009 =

$\underbrace{(x-k) + (x-k+1) + \cdots + (x+k-1) + (x+k)}_{n\ terms}$,

where x and k are integers such that k < x.

n = 1 + 2 k

therefore, 2009 = (1 + 2k)x

Since 2009 = 7 × 7 × 41, to find the largest value of n we need the largest value of k, and hence the smallest value of x.

Trying x = 1, 1 + 2k = 2009, therefore k = 1004 > x (rejected)

x = 7, 1 + 2k = 287, therefore k = 143 > x (rejected)

x = 41, 1 + 2k = 49, therefore k = 24 < x

Hence, n = 49.

Q 13: 135

Let the HCF of x and 15 be k and let x = ka, 15 = kb where gcd (a,b) = 1. We have LCM of x and 15 = kab. From information given, kab – k = 120. Clearly gcd (ab – 1, b) =1, so gcd (kab – k, kb) = k. Therefore, gcd (120, 15) = k. Thus, k = 15.

Hence, b = 1, a = 9 and x = 15 × 9 = 135

Q14: 9

First we note that the last digit of 2009^{2009} is the same as that of 9^{2009}.

Now the last digits of $9^1, 9^2, 9^3, 9^4, 9^5$ are 9, 1, 9, 1, 9 respectively and it should be clear that for 9^n, the last digit is 1 when n is even and the last digit is 9 when n is odd. Since 2009 is odd, the last digit of 2009^{2009} is 9.